Linear Algebra in the vector space of intervals $\overline{\mathbb{IR}}$

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Abstract

In a previous paper, we have given an algebraic model to the set of intervals. Here, we apply this model in a linear frame. We define a notion of diagonalization of square matrices whose coefficients are intervals. But in this case, with respect to the real case, a matrix of order n could have more than n eigenvalues (the set of intervals is not factorial). We consider a notion of central eigenvalues permits to describe criterium of diagonalization. As application, we define a notion of Exponential mapping.

1 The associative algebra $\overline{\mathbb{IR}}$

In [1], we have given a representation of the set of intervals in terms of associative algebra. More precisely, we define on the set $\mathbb{IR}$ of intervals of $\mathbb{R}$ a $\mathbb{R}$-vector space structure. Next we embed $\mathbb{IR}$ in a 4-dimensional associative algebra. This embedding permits to describe a unique distributive multiplication which contains all the possible results of the usual product of intervals and the monotony property is always conserved. Moreover, this new product is minimal with respect the distributivity and the monotony properties.

In this section, we present briefly this construction (for more details, see [1]).

1.1 Vector space structure on $\mathbb{IR}$

Let $\mathbb{IR}$ be the set of intervals of $\mathbb{R}$, that is

$$\mathbb{IR} = \{[a, b], a, b \in \mathbb{R},\ a \leq b\}.$$

This set is provided with a semi-group structure that we can complete as follow: we consider the equivalence relation on $\mathbb{IR} \times \mathbb{IR}$:

$$(X, Y) \sim (Z, T) \iff X + T = Y + Z$$

for all $X, Y, Z, T \in \mathbb{IR}$. The quotient set is denoted by $\overline{\mathbb{IR}}$. The addition of intervals is compatible with this equivalence relation :

$$\overline{(X, Y)} + \overline{(z, t)} = \overline{(x + z, y + t)}$$

where $\overline{(x, y)}$ is the equivalence class of (X, Y). The unit is $\overline{0} = \{(X, X), X \in \mathbb{IR}\}$ and each element has an inverse

$$\diagdown\overline{(X, Y)} = \overline{(Y, X)}.$$

Then $(\overline{\mathbb{IR}}, +)$ is a commutative group. We prove also in [1], that any equivalence class admits a canonical representant of type $(K, 0)$ or $(0, K)$ or (a, a), with $K \in \mathbb{IR}$ and $a = [a, a]$, $a \in \mathbb{R}$. We provides the group $(\overline{\mathbb{IR}}, +)$ with a real vector space structure, the external product being given by

$$\begin{cases} \alpha \cdot \overline{(K, 0)} = \overline{(\alpha K, 0)} \\ \alpha \cdot \overline{(0, K)} = \overline{(0, \alpha K)} \end{cases}$$

if $\alpha > 0$. If $\alpha < 0$ we put $\beta = -\alpha$ and

$$\begin{cases} \alpha \cdot \overline{(K, 0)} = \overline{(0, \beta K)} \\ \alpha \cdot \overline{(0, K)} = \overline{(\beta K, 0)} \end{cases} .$$

The triplet $(\overline{\mathbb{IR}}, +, \cdot)$ is a real vector space.

Remark. To simplify notations, we write $(K, 0)$ or $(0, K)$ in place of $\overline{(K, 0)}$ or $\overline{(0, K)}$.

1.2 The associative algebra $\mathcal{A}_4$

Recall that by an algebra we mean a real vector space with an associative ring structure. Consider the 4-dimensional associative algebra whose product in a basis $\{e_1, e_2, e_3, e_4\}$ is given by

	e_1	e_2	e_3	e_4
e_1	e_1	0	0	e_4
e_2	0	e_2	e_3	0
e_3	0	e_3	e_2	0
e_4	e_4	0	0	e_1

The unit is the vector $e_1 + e_2$. This algebra is a direct sum of two ideals: $\mathcal{A}_4 = I_1 + I_2$ where I_1 is generated by e_1 and e_4 and I_2 is generated by e_2 and e_3. It is not an integral domain, that is, we have divisors of 0. For example $e_1 \cdot e_2 = 0$. The ring $\mathcal{A}_4$ is principal that is every ideal is generated by one element. The cartesian expression of this product is, for $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ in $\mathcal{A}_4$:

$$x \cdot y = (x_1 y_1 + x_4 y_4, x_2 y_2 + x_3 y_3, x_3 y_2 + x_2 y_3, x_4 y_1 + x_1 y_4).$$

The multiplicative group $\mathcal{A}_4^*$ of invertible elements is the set of elements $x = (x_1, x_2, x_3, x_4)$ such that

$$\begin{cases} x_4 \neq \pm x_1, \\ x_3 \neq \pm x_2. \end{cases}$$

If $x \in \mathcal{A}_4^*$ we have:

$$x^{-1} = \left(\frac{x_1}{x_1^2 - x_4^2}, \frac{x_2}{x_2^2 - x_3^2}, \frac{x_3}{x_2^2 - x_3^2}, \frac{x_4}{x_1^2 - x_4^2} \right).$$

1.3 A product in $\overline{\mathbb{IR}}$

We define a correspondence between $\mathbb{IR}$ and $\mathcal{A}$. Let φ be the map

$$\varphi : \overline{\mathbb{IR}} \longrightarrow \mathcal{A}_4$$

given by

$$\varphi(K,0) = \begin{cases} (x_1, x_2, 0, 0) \text{ if } x_1, x_2 \geq 0, \\ (0, x_2, -x_1, 0) \text{ if } x_1 \leq 0 \text{ and } x_2 \geq 0, \\ (0, 0, -x_1, -x_2) \text{ if } x_1, x_2 \leq 0, \end{cases}$$

and

$$\overline{\varphi}(0, K) = -\overline{\varphi}(K, 0).$$

for any $K = [x_1, x_2]$. Then we have

$$\overline{\varphi}(0, K) \quad = \quad \begin{cases} -x_1, -x_2, 0, 0) \text{ if } x_1 \geq 0, \\ (0, -x_2, -x_1, 0) \text{ if } x_1 x_2 \leq 0, \\ (0, 0, -x_1, -x_2) \text{ if } x_2 \leq 0. \end{cases}$$

Thus the image of $\overline{\overline{\mathbb{IR}}}$ in $\mathcal{A}_4$ is constituted of the elements

$$\begin{cases} (x_1, x_2, 0, 0) \text{ with } 0 \leq x_1 \leq x_2 \text{ which corresponds to } ([x_1, x_2], 0), \\ (0, x_2, -x_1, 0) \text{ with } x_1 \leq 0 \leq x_2 \text{ which corresponds to } ([x_1, x_2], 0), \\ (0, 0, -x_1, -x_2) \text{ with } x_1 \leq x_2 \leq 0 \text{ which corresponds to } ([x_1, x_2], 0), \\ (-x_1, -x_2, 0, 0) \text{ with } 0 \leq x_1 \leq x_2 \text{ which corresponds to } (0, [x_1, x_2]), \\ (0, -x_2, x_1, 0) \text{ with } x_1 \leq 0 \leq x_2 \text{ which corresponds to } (0, [x_1, x_2]), \\ (0, 0, x_1, x_2) \text{ with } x_1 \leq x_2 \leq 0 \text{ which corresponds to } (0, [x_1, x_2]). \end{cases}$$

The map $\overline{\varphi} : \overline{\mathbb{IR}} \longrightarrow \mathcal{A}_4$ is not linear. We introduce in $\mathcal{A}_4$ the following equivalence relation $\mathcal{R}$ given by

$$(x_1, x_2, x_3, x_4) \sim (y_1, y_2, y_3, y_4) \Longleftrightarrow \begin{cases} x_1 - y_1 = x_3 - y_3, \\ x_2 - y_2 = x_4 - y_4 \end{cases}$$

and consider the map

$$\overline{\overline{\varphi}} : \overline{\mathbb{IR}} \longrightarrow \overline{\mathcal{A}_4} = \frac{\mathcal{A}_4}{\mathcal{R}}$$

given by $\overline{\overline{\varphi}} = \Pi \circ \overline{\varphi}$ where Π is a canonical projection. This map is surjective. In fact we have the correspondence

- $x_1 - x_3 \geq 0, x_2 - x_4 \geq 0, x_1 - x_3 \leq x_2 - x_4$

$$(x_1, x_2, x_3, x_4) \sim (x_1 - x_3, x_2 - x_4, 0, 0) = \overline{\varphi}([x_1 - x_3, x_2 - x_4], 0).$$

- $x_1 - x_3 \geq 0, x_2 - x_4 \geq 0, x_1 - x_3 \geq x_2 - x_4$

$$(x_1, x_2, x_3, x_4) \sim (0, 0, x_3 - x_1, x_4 - x_2) = \overline{\varphi}(0, [x_3 - x_1, x_4 - x_2]).$$

- $x_1 - x_3 \geq 0, x_2 - x_4 \leq 0$

$$(x_1, x_2, x_3, x_4) \sim (0, x_2 - x_4, x_3 - x_1, , 0) = \overline{\varphi}(0, [x_3 - x_1, x_4 - x_2]).$$

- $x_1 - x_3 \leq 0, x_2 - x_4 \geq 0$

$$(x_1, x_2, x_3, x_4) \sim (0, x_2 - x_4, x_3 - x_1, , 0) = \overline{\varphi}([x_3 - x_1, x_2 - x_4], 0).$$

3

- $x_1 - x_3 \leq 0$, $x_2 - x_4 \leq 0$, $x_1 - x_3 \geq x_2 - x_4$

$$(x_1, x_2, x_3, x_4) \sim (x_1 - x_3, x_2 - x_4, 0, 0) = \overline{\varphi}(0, [x_3 - x_1, x_4 - x_2]).$$

- $x_1 - x_3 \leq 0$, $x_2 - x_4 \leq 0$, $x_1 - x_3 \leq x_2 - x_4 (x_1, x_2, x_3, x_4) \sim (0, 0, x_3 - x_1, x_4 - x_2) = \overline{\varphi}([x_3 - x_1, x_2 - x_4], 0)$.

This correspondence defines a map

$$\psi : \overline{\mathcal{A}_4} \longrightarrow \overline{\overline{\mathbb{R}}}.$$

In the following, to simplify notation, we write $\overline{\varphi}$ instead of $\overline{\overline{\varphi}}$.

Definition 1 *For any $\mathcal{X}, \mathcal{X}' \in \overline{\overline{\mathbb{R}}}$, we put*

$$\mathcal{X} \bullet \mathcal{X}' = \psi(\overline{\overline{\varphi}(\mathcal{X})} \bullet \overline{\overline{\varphi}(\mathcal{X}')}).$$

This multiplication is distributive with respect the the addition. In fact

$$(\mathcal{X}_1 + \mathcal{X}_2) \bullet \mathcal{X}' = \psi(\overline{\overline{\varphi}(\mathcal{X}_1 + \mathcal{X}_2)} \bullet \overline{\overline{\varphi}(\mathcal{X}')}).$$

Suppose that $\overline{\varphi}(\mathcal{X}_1 + \mathcal{X}_2) \neq \overline{\varphi}(\mathcal{X}_1) + \overline{\varphi}(\mathcal{X}_2)$. In this case this means that $\overline{\varphi}(\mathcal{X}_1) + \overline{\varphi}(\mathcal{X}_2) \notin Im\overline{\varphi}$. But by construction $\overline{\varphi}(\mathcal{X}_1 + \mathcal{X}_2) \in Im\overline{\varphi}$ and this coincides with $\overline{\varphi}(\mathcal{X}_1 + \mathcal{X}_2)$. For numerical application of this product, see ([**?**, GN-R].

2 The module $gl(n, \overline{\overline{\mathbb{R}}})$

Let $gl(n, \overline{\overline{\mathbb{R}}})$ be the set of square matrices of order n whose elements are in $\overline{\overline{\mathbb{R}}}$. A matrice of $gl(n, \overline{\overline{\mathbb{R}}})$ is denoted by

$$A = (\mathcal{X}_{ij})_{i,j=1,\cdots,n}$$

with $\mathcal{X}_{ij} = (K_{ij}, 0)$ or $(0, K_{ij})$. It is clear that $gl(n, \overline{\overline{\mathbb{R}}})$ is a real vector space. We define a product on it puting

$$A \cdot B = (\mathcal{X}_{ij}) \cdot (Y_{ij}) = (Z_{ij})$$

with $Z_{ij} = \sum_{k=1}^{n} \mathcal{X}_{ik} \cdot Y_{ik}$. This last product being the associative product on $\overline{\overline{\mathbb{R}}}$. Thus $gl(n, \overline{\overline{\mathbb{R}}})$ is an associative algebra.

Définition 1 *A matrice $A \in gl(n, \overline{\overline{\mathbb{R}}})$ is called inversible if its determinant, computed by the Cramer rule, is an inversible element in $\overline{\overline{\mathbb{R}}}$.*

Recall that the group $\overline{\overline{\mathbb{R}}}$ of inversible elements contain

$$\mathcal{X}_i = (K_i, 0) \text{ or } (0, K_i)$$

with $0 \notin K_i$. To compute the determinant, we use the classical formula of Cramer.

Example 1. Let us consider the matrix

$$M = \begin{pmatrix} [1, 2] & [-1.3] \\ [-1, 3] & [1, 2] \end{pmatrix}$$

4

Thus

$$
\begin{aligned}
\det B_1 &= ([1,2],0)([1,2],0) \setminus ([-1,3],0)([-1,3],0) \\
&= ([1,4],0) \setminus ([-3,9],0) \\
&= ([0,[-4,5]) \\
&= \setminus([-4,5],0).
\end{aligned}
$$

As $([-4,5],0)$ is not an inversible element of $\overline{\mathbb{IR}}$, the matrix B_1 is not inversible.

Example 2. Now if

$$
B_2 = \left(\begin{array}{cc} [1,2] & [-1.3] \\ [-1,3] & [1,7] \end{array} \right)
$$

then, by the similar computation, we obtain

$$
\det B_2 = \setminus([-7,-4],0)
$$

and B_2 is invertible.

Definition 2 *If A is an invertible matrix on $gl(n,\overline{\mathbb{IR}})$, the inverse matrix A^{-1} of A is given by*

$$
A \cdot A^{-1} = Id
$$

where

$$
Id = \left(\begin{array}{cccc} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{array} \right)
$$

with $1 = ([1,1],0)$ and $0 = ([0,0],0)$.

The determination of A^{-1} can be computed using the classical rules.

Example. If we consider the invertible matrix B_2, we obtain

$$
B_2^{-1} = [\frac{1}{7}, \frac{1}{4}] \left(\begin{array}{cc} [1,7] & \setminus[-1.3] \\ \setminus[-1,3] & [1,2] \end{array} \right).
$$

Let us verify that $B_2 B_2^{-1} = Id$. Using the product on $\overline{\mathbb{IR}}$ we obtain

$$
B_2 B_2^{-1} = [\frac{1}{7}, \frac{1}{4}] \left(\begin{array}{cc} [1,2] & [-1.3] \\ [-1,3] & [1,7] \end{array} \right) \cdot \left(\begin{array}{cc} [1,7] & \setminus[-1.3] \\ \setminus[-1,3] & [1,2] \end{array} \right).
$$

The coefficient in place $(1,1)$ is

$$
a_{11} = [\frac{1}{7}, \frac{1}{4}]([1,2][1,7] + [-1.3](\setminus[-1,3])).
$$

¿From the definition of the product (see section 1), this element is

$$
\begin{aligned}
a_{11} &= (\frac{1}{7}, \frac{1}{4}, 0, 0)((1, 2, 0, 0)(1, 7, 0, 0) - (0, 3, 1, 0)(0, 3, 1, 0)) \\
&= (\frac{1}{7}, \frac{1}{4}, 0, 0)((1, 14, 0, 0) - (0, 10, 6, 0)) \\
&= (\frac{1}{7}, \frac{1}{4}, 0, 0)(1, 4, -6, 0) \\
&= (\frac{1}{7}, \frac{1}{4}, 0, 0)(7, 4, 0, 0) \\
&= (1, 1, 0, 0)
\end{aligned}
$$

which corresponds to $[1, 1]$. Similarly we have $a_{12} = a_{21} = (0, 0, 0, 0)$ and $a_{22} = (1, 1, 0, 0)$. Thus $B_2 B_2^{-1} = Id$.

3 Diagonalization

3.1 Eigenvalues and central eigenvalues

Let A be in $gl(n, \overline{\overline{\mathbb{R}}})$. An eigenvalue of A is an element $\mathcal{X} \in \overline{\overline{\mathbb{R}}}$ such that there exists a vector $\mathcal{V} \neq 0 \in \overline{\overline{\mathbb{R}}}^n$ with

$$
A \cdot^t \mathcal{V} = \mathcal{X} \cdot^t \mathcal{V}.
$$

Thus $\mathcal{X}$ is a root of the characteristical polynomial with coefficients in the ring $\overline{\overline{\mathbb{R}}}$

$$
C_A(\mathcal{X}) = \det(A - \mathcal{X}I) = 0.
$$

Example. Let

$$
B_3 = \left(\begin{array}{cc} [1, 2] & [1, 2] \\ [1, 3] & [2, 5] \end{array} \right).
$$

We have

$$
B_3 - \mathcal{X}I = \left(\begin{array}{cc} [1, 2] \smallsetminus \mathcal{X} & [1, 2] \\ [1, 3] & [2, 5] \smallsetminus \mathcal{X} \end{array} \right)
$$

and

$$
\begin{aligned}
\det(B_3 - \mathcal{X}I) &= ([1, 2] \smallsetminus \mathcal{X})([2, 5] \smallsetminus \mathcal{X}) - [1, 3][1, 2] \\
&= [2, 10] - \mathcal{X}[2, 5] - \mathcal{X}[1, 2] + (\smallsetminus \mathcal{X})(\smallsetminus \mathcal{X}) - [1, 6] \\
&= (\smallsetminus \mathcal{X})(\smallsetminus \mathcal{X}) - \mathcal{X}[3, 7] + [1, 4].
\end{aligned}
$$

Let $\mathcal{X} = ([x, y], 0)$. It is represented in $\mathcal{A}_4$ by $(x, y, 0, 0)$ or $(0, y, x, 0)$ or $(0, 0, x, y) = -(x, y, 0, 0)$.

First case: $\det(B_3 - \mathcal{X}I) = (x^2, y^2, 0, 0) - (3x, 7y, 0, 0) + (1, 4, 0, 0) = (x^2 - 3x + 1, y^2 - 7y + 4, 0, 0)$. Then $\det(B_3 - \mathcal{X}I) = 0$ implies

$$
\left\{ \begin{array}{l} x^2 - 3x + 1 = 0, \\ y^2 - 7y + 4 = 0, \end{array} \right.
$$

6

that is

$$\begin{cases} x = \dfrac{3 \pm \sqrt{5}}{2}, \\ y = \dfrac{7 \pm \sqrt{33}}{2}. \end{cases}$$

We obtain

$$\begin{cases} \mathcal{X}_1 = ([\dfrac{3 + \sqrt{5}}{2}, \dfrac{7 + \sqrt{33}}{2}], 0), \\ \mathcal{X}_2 = ([\dfrac{3 - \sqrt{5}}{2}, \dfrac{7 + \sqrt{33}}{2}], 0), \\ \mathcal{X}_3 = ([\dfrac{3 - \sqrt{5}}{2}, \dfrac{7 - \sqrt{33}}{2}], 0). \end{cases}$$

Second case: $\det(B_3 - \mathcal{X}I) = (0, y^2 + x^2, 2xy, 0) - (0, 7y, 7x, 0) + (1, 4, 0, 0) = (1, y^2 + x^2 - 7y + 4, 2xy - 7x, 0)$. Then $\det(B_3 - \mathcal{X}I) = 0$ implies

$$\begin{cases} 1 - 2xy + 7x = 0, \\ y^2 + x^2 - 7y + 4 = 0. \end{cases}$$

This gives

$$4y^4 - 56y^3 + 261y^2 - 455y + 197 = 0.$$

We have the following solutions

$$(x; y) = \{(-2, 8; 3, 32), (2, 9; 3, 67), (-0, 17; 0, 63), (0, 17; 6, 37)\}.$$

We obtain the eigenvalues

$$\begin{cases} \mathcal{X}_4 = ([-2, 8, \ 3.32], 0), \\ \mathcal{X}_5 = ([-0.17, \ 0.63], 0). \end{cases}$$

Third case: $\det(B_3 - \mathcal{X}I) = (x^2, y^2, 0, 0) + (3x, 7y, 0, 0) + (1, 4, 0, 0) = (x^2 + 3x + 1, y^2 + 7y + 4, 0, 0)$. Then $\det(B_3 - \mathcal{X}I) = 0$ implies

$$\begin{cases} x = \dfrac{-3 \pm \sqrt{5}}{2}, \\ y = \dfrac{-7 \pm \sqrt{33}}{2}, \end{cases}$$

then

$$\mathcal{X}_6 = ([\dfrac{-3 - \sqrt{5}}{2}, \dfrac{-7 + \sqrt{33}}{2}], 0).$$

We obtain six eigenvalues.

Remark. To compute the interval-eigenvalues of a matrix A, we have to find the roots of the characteristical polynomial of A. But this polynomial is with coefficients in $\overline{\mathbb{IR}}$ (or $\mathcal{A}_4$) and this set is not a field neither a factorial ring. Then it is natural to meet some special results (e.g if we consider the second degree polynomial $X^2 - 1$ with coefficients in $\frac{\mathbb{Z}}{8\mathbb{Z}}$ which is not factorial, it admits four roots,$1, 3, 5, 7$.) In our example we finds 6 roots. Now if we consider the real matrix whose coefficients are the centers of interval-coefficients of B_3, that is

$$c_{B_3} = \begin{pmatrix} 1.5 & 1.5 \\ 2 & 3.5 \end{pmatrix}$$

then the eigenvalues of c_{B_3} are 4.5 and 0.5 which are closed to the center of $\mathcal{X}_1$ and $\mathcal{X}_3$. We call these eigenvalues, the central eigenvalues.

Definition 3 *Let A be a matrix in $gl(n, \overline{\mathbb{IR}})$. Lat A_c be the real matrix whose elements are the center of the intervals of A. We say that an eigenvalue of A is a central eigenvalue if its center is (close to) an eigenvalue of A_c.*

Remark. The determination of negative eigenvalues that is of type $(0, K)$ is similar. Nevertheless we have to consider only matrices with positive entries thus we studies only the positive eigenvalues. The negative eigenvalues do not correspond to physical entities.

3.2 Eigenvectors, eigenspaces

Now we will look the problem of reduction of an interval matrix. Recall that the characteristical polynomial is with coefficient in a non factorial ring. This is the bigest change with respect the classical real linear algebra.

Définition 2 *Let A a square matrix with coeffcients in $\overline{\mathbb{IR}}$. If $\mathcal{X}$ is an eigenvalue of A, then every vector $\mathcal{V} \in \overline{\mathbb{IR}}^n$ satisfying $A^t \mathcal{V} = \mathcal{X}^t \mathcal{V}$ is an eigenvector associated with $\mathcal{X}$.*

Let $E_{\mathcal{X}}$ be the set
$$E_{\mathcal{X}} = \{ \mathcal{V} \in \overline{\mathbb{IR}}^n \text{ such that } A^t \mathcal{V} = \mathcal{X}^t \mathcal{V} \}.$$
Then $E_{\mathcal{X}}$ is a $\mathbb{R}$-subspace of $\overline{\mathbb{IR}}^n$ where n is the order of the matrix A. It is also a $\overline{\mathbb{IR}}$ submodule of $\overline{\mathbb{IR}}^n$.

Proposition 1 *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two distinguish eigenvalues of A. Then $E_{\mathcal{X}_1} \cap E_{\mathcal{X}_2} = \{0\}$.*

Proof. Let $\mathcal{V}$ be in $E_{\mathcal{X}_1} \cap E_{\mathcal{X}_2}$. We have

$$
\begin{aligned}
A^t \mathcal{V} &= \mathcal{X}_1 \mathcal{V}, \\
A^t \mathcal{V} &= \mathcal{X}_2 \mathcal{V}.
\end{aligned}
$$

This $\mathcal{X}_1 \mathcal{V} \smallsetminus \mathcal{X}_2 \mathcal{V} = (\mathcal{X}_1 \smallsetminus \mathcal{X}_2) \mathcal{V} = 0$. As $\overline{\mathbb{IR}}$ is without zero divisor, we have $\mathcal{X}_1 \smallsetminus \mathcal{X}_2 = 0$ or $\mathcal{V} = 0$. We deduce $E_{\mathcal{X}_1} \cap E_{\mathcal{X}_2} = \{0\}$.

Proposition 2 *Let $C_A(\mathcal{X})$ be the characteristical polynomial of A. If the real polynomial $C_{C_A}(\mathcal{X})$ associated with the central matrix of A is a product of factor of degree 1, then $C_A(\mathcal{X})$ admits a factorization on $\overline{\mathbb{IR}}$*

We have seen that $C_A(\mathcal{X})$ can be have more than degree($C_A(\mathcal{X})$) roots. If $\mathcal{X}_1, \cdots, \mathcal{X}_n$ are the central roots, we have the decomposition
$$C_A(\mathcal{X}) = a_n \prod_{i=1}^{n} (\mathcal{X} \smallsetminus \mathcal{X}_i).$$

Example. If we consider the matrix
$$B_3 = \begin{pmatrix} [1,2] & [1,2] \\ [1,3] & [2,5] \end{pmatrix}.$$

8

then $C_{B_3}(\mathcal{X})$ admits $\mathcal{X}_1, \cdots, \mathcal{X}_6$ as positive roots. The central eigenvalues are $\mathcal{X}_1$ and $\mathcal{X}_3$ and we have

$$\det(B_3 \smallsetminus \mathcal{X}I) = (\mathcal{X} \smallsetminus \mathcal{X}_1)(\mathcal{X} \smallsetminus \mathcal{X}_3).$$

If we consider the roots $\mathcal{X}_2 = ([\dfrac{3 - \sqrt{5}}{2}, \dfrac{7 + \sqrt{33}}{2}], 0)$, and if we assume that $C_{B_{3}s}(\mathcal{X}) = (\mathcal{X} \smallsetminus \mathcal{X}_2)(\mathcal{X} \smallsetminus Y)$, we obtain

$$Y = (3, 7, \dfrac{3 - \sqrt{5}}{2}, \dfrac{7 + \sqrt{33}}{2}) = (\dfrac{3 + \sqrt{5}}{2}, \dfrac{7 - \sqrt{33}}{2}, 0, 0)$$

which does not correspond to a positive eigenvalue.

Theorem 1 *For any n-ule of roots* $(\mathcal{X}_1, \cdots, \mathcal{X}_n)$ *such that* $C_A(\mathcal{X}) = a_n \displaystyle\prod_{i=1}^{n} (\mathcal{X} \smallsetminus \mathcal{X}_i)$, *and if for any* $i = 1, \cdots, n$ *the dimension of* $E_{\mathcal{X}_i}$ *coincides with the multiplicity of* $\mathcal{X}_i$, *then we have the vectorial decomposition* $\overline{\mathbb{IR}}^n = \oplus_{i \in I} E_{\mathcal{X}_i}$ *where the roots* $\mathcal{X}_i, i \in I$ *are pairwise distinguish.*

Example. Let us compute the eigenspaces of B_3 associated to the central eigenvalues.

- $\mathcal{X}_1 = ([\dfrac{3 + \sqrt{5}}{2}, \dfrac{7 + \sqrt{33}}{2}], 0)$.

 Let $V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \in \overline{\mathbb{IR}}$. Then

$$(A - \mathcal{X}_1 I) \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = 0$$

 is equivalent to

$$\begin{pmatrix} (0, [\dfrac{1 + \sqrt{5}}{2}, \dfrac{3 + \sqrt{33}}{2}])V_1 & ([1,2], 0)V_2 \\ ([1.3], 0)V_1 & (0, [\dfrac{-1 + \sqrt{5}}{2}, \dfrac{-3 + \sqrt{33}}{2}])V_2 \end{pmatrix} = 0$$

 that is

$$\begin{cases} \smallsetminus[\dfrac{1 + \sqrt{5}}{2}, \dfrac{3 + \sqrt{33}}{2}]V_1 + [1,2]V_2 = 0, \\ [1.3]V_1 \smallsetminus [\dfrac{-1 + \sqrt{5}}{2}, \dfrac{-3 + \sqrt{33}}{2}]V_2 = 0. \end{cases}$$

 This gives

$$V_2 = \dfrac{[\dfrac{1 + \sqrt{5}}{2}, \dfrac{3 + \sqrt{33}}{2}]V_1}{[1, 2]}.$$

 If we choose $V_1 = ([1, 1], 0)$ we have

$$\begin{aligned} V_2 &= \dfrac{[\dfrac{1 + \sqrt{5}}{2}, \dfrac{3 + \sqrt{33}}{2}]}{[1,2]} \\ &= [\dfrac{-1 + \sqrt{5}}{2}, \dfrac{-3 + \sqrt{33}}{2}] \bullet (\smallsetminus[-1, \dfrac{-1}{2}]) \\ &= \smallsetminus([-\dfrac{3 + \sqrt{33}}{2}, -\dfrac{1 + \sqrt{5}}{4}]) \end{aligned}$$

9

Thus the $\mathcal{X}_1$-eigenvectors are of the form

$$V = \begin{pmatrix} ([1,1],0) \\ \searrow([-\frac{3+\sqrt{33}}{2}, -\frac{1+\sqrt{5}}{4}],0) \end{pmatrix}.$$

Remark. We can choose V_1 such that all tje coordinatez of V are positive. For example if $V_1 = [1,2]$

then $V = \begin{pmatrix} ([1,2],0) \\ ([\dfrac{1+\sqrt{5}}{2}, \dfrac{3+\sqrt{33}}{2}],0) \end{pmatrix}$

- $\mathcal{X}_3 = ([\dfrac{3-\sqrt{5}}{2}, \dfrac{7-\sqrt{33}}{2}],0).$

  Let $V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \in \overline{\mathrm{I\!R}}$. Then

  $$(A - \mathcal{X}_1 I)\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = 0$$

  is equivalent to

  $$\begin{pmatrix} ([\dfrac{-1+\sqrt{5}}{2}, \dfrac{-3+\sqrt{33}}{2}],0)V_1 & ([1,2],0)V_2 \\ ([1.3],0)V_1 & ([\dfrac{1+\sqrt{5}}{2}, \dfrac{3+\sqrt{33}}{2}],0)V_2 \end{pmatrix} = 0$$

  that is

  $$\begin{cases} [\dfrac{-1+\sqrt{5}}{2}, \dfrac{-3+\sqrt{33}}{2}]V_1 + [1,2]V_2 = 0, \\ [1.3]V_1 + [\dfrac{1+\sqrt{5}}{2}, \dfrac{-3+\sqrt{33}}{2}]V_2 = 0. \end{cases}$$

  This gives

  $$V_2 = \dfrac{\searrow([\dfrac{-1+\sqrt{5}}{2}, \dfrac{-3+\sqrt{33}}{2}])V_1}{[1,2]}.$$

  If we choose $V_1 = ([1,1],0)$ we have

  $$\begin{aligned} V_2 &= \dfrac{\searrow[\dfrac{-1+\sqrt{5}}{2}, \dfrac{-3+\sqrt{33}}{2}]}{[1,2]} \\ &= \searrow[\dfrac{-1+\sqrt{5}}{2}, \dfrac{-3+\sqrt{33}}{2}] \bullet (\searrow[-1, \tfrac{-1}{2}]) \\ &= ([\tfrac{3-\sqrt{33}}{2}, \tfrac{1-\sqrt{5}}{4}]) \end{aligned}$$

  Thus the $\mathcal{X}_3$-eigenvectors are of the form

  $$V = \begin{pmatrix} ([1,1],0) \\ ([\tfrac{3-\sqrt{33}}{2}, \tfrac{1-\sqrt{5}}{4}],0) \end{pmatrix}.$$

10

4 The Exponential map

We define the exponential map

$$Exp : gl(n, \overline{\mathbb{IR}}) \longrightarrow gl(n, \overline{\mathbb{IR}})$$

in a classical way by series expansions. If the matrix A is diagonalizable, then

$$D = P^{-1}AP$$

is diagonal and $Exp(A)$ is a diagonal matrix whose diagonal element are the exponential of the eigenvalues.

Example. Let

$$B_3 = \left(\begin{array}{cc} [1,2] & [1,2] \\ [1,3] & [2,5] \end{array} \right).$$

The central eigenvalues are

$$\begin{cases} \mathcal{X}_1 = ([\dfrac{3+\sqrt{5}}{2}, \dfrac{7+\sqrt{33}}{2}], 0), \\ \mathcal{X}_3 = ([\dfrac{3-\sqrt{5}}{2}, \dfrac{7-\sqrt{33}}{2}], 0). \end{cases}$$

and we have

$$D = \left(\begin{array}{cc} \mathcal{X}_1 & 0 \\ 0 & \mathcal{X}_3 \end{array} \right)$$

with

$$P = \left(\begin{array}{cc} ([1,1], 0) & ([1,1], 0) \\ \searrow([-\frac{3+\sqrt{33}}{2}, -\frac{1+\sqrt{5}}{4}], 0) & ([\frac{3-\sqrt{33}}{2}, \frac{1-\sqrt{5}}{4}], 0) \end{array} \right).$$

We deduce

$$Exp(B_3) = P. \left(\begin{array}{cc} ([exp(\dfrac{3+\sqrt{5}}{2}), exp(\dfrac{7+\sqrt{33}}{2}], 0)) & 0 \\ 0 & ([exp(\dfrac{3-\sqrt{5}}{2}), exp(\dfrac{7-\sqrt{33}}{2}], 0)) \end{array} \right).P^{-1}$$

In a forthcomming paper, we apply this calculus to solve linear differential system.

References

[1] Goze Nicolas, Remm Elisabeth. An algebraic approach to the set of intervals. arXiv math 0809.5150 (2008)